Journal of Computational and Applied Mathematics 48 (1993) 157–165
North-Holland

CAM 1405

CONVODE: a REDUCE package for solving differential equations

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Received 16 February 1992

Abstract

Moussiaux, A., CONVODE: a REDUCE package for solving differential equations, Journal of Computational and Applied Mathematics 48 (1993) 157–165.

The package CONVODE (CONVersion of Ordinary Differential Equations) is written in order to study differential equations and systems of differential equations.

Keywords: Differential equations; REDUCE; algebraic programming.

1. Introduction

The package CONVODE (CONVersion of Ordinary Differential Equations) is written in order to study differential equations and systems of differential equations. In this paper, differential equations whose solution can be obtained using "elementary" methods of integration are investigated. In the literature on differential equations which can be solved in such a way (with elementary methods), examples are available. Most of the books on ODEs are practically written based on the same framework.

1.1. First-order differential equations

When the problem involves only one differential equation of first order, at least thirty cases have been programmed. CONVODE starts by asking itself the question if the proposed equation is *linear* or not. If it is linear, is it homogeneous or nonhomogeneous, are the initial conditions given? If the equation is linear, no problem, CONVODE gives you the solution; otherwise, another series of approaches begins, with the aim of finding a particular property allowing its integration. Here are some investigated cases.

- Does solving the equation lead to the computation of a simple primitive?
- Is the equation *exact*?

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• Is the equation *homogeneous*?
• Is the equation *separable*?
• Is your equation perhaps of *Bernoulli* type?
• Is it possible to find a classical *integrating factor*? For example, is it possible to find α and β in such a way that $x^\alpha y^\beta$ becomes an integrating factor? Would a transformation of the form $y = z^\alpha$ be able to manage the problem? Could we hope that a change of variable of the type $x = u + h$, $y = v + k$, with h and k constants, leads to a *homogeneous equation*? And so on.
• An effort has been made in the case of *Riccati* equations. In the general case, the *Riccati* equation is untractable, except in particular cases where explicit solutions can be found. In such a case, CONVODE provides you with the solution (with or without initial conditions).
• The *Lagrange–Clairaut* equations (which are not linear with respect to the derivative) are investigated by CONVODE exactly as we would do it by hand. A parametrical solution is proposed when possible, as well as the singular solution.
• For equations which do not depend on x or y, a parametric solution is also proposed.
• The study of differential equations not solved with respect to the derivative is also done by CONVODE. If the equation may be solved (with respect to the derivative), each solution is studied again by the program. The list RESPONSE collects all solutions that have been found.

CONVODE is far from doing everything, but one of its characteristics is that it is a very wordy program and that it keeps its user aware of its process. If in the answer RESPONSE you only have the comment

"Sorry, I didn't find, but I slogged hard",

you already know that your equation is not *linear*, that it is not reducing to a *simple integral*, that it is not *exact*, not *homogeneous*, that there is no *classical integrating factor*, that it is neither *Bernouilli* nor *Riccati*. You know that it is not *Lagrange–Clairaut* and that a parametrical solution could not have been obtained, etc. The only thing to do is to invert the role of the independent and dependent variables in the first-order equation, which can sometimes bring you a nice surprise. (It is evident that very often it does not work, but sometimes!)

1.2. Systems of first-order differential equations

In the case of *systems* of equations, CONVODE can only solve the systems with constant coefficients. CONVODE gives the characteristic equation, its solutions and their multiplicity.

If the system is *nonhomogeneous*, a particular solution is calculated in such a way to give the general solution. Depending on whether the initial conditions are fixed or not, you will find in RESPONSE the adequate solution or the solution in terms of arbitrary constants.

1.3. The systems of differential equations of order greater than one

CONVODE was first made to solve first-order systems. If the system is of *order greater than one*, we know that it is always possible to transform it to a first-order system which uses a greater number of independent variables.

So, by adding to the ODE procedure which solves the first-order systems a CONVersion procedure, the problem is solved. We only have, at the end, to go back to the initial variables. A particular simple and important case is the one where we have only one linear differential equation with constant coefficients (homogeneous, nonhomogeneous, with or without initial conditions). For example, a second-order differential equation is replaced by a system of two first-order differential equations. CONVODE always gives the solutions of the first-order equivalent system and of course the solutions of the initial system.

This conversion phase can of course be criticized; is it a waste of time?, is there something to be gained?, etc. It is important for *degenerated* systems, which will be treated later.

1.4. The equations of Euler and the ones which can be reduced to the Euler form

CONVODE can solve differential equations with constant coefficients if they are linear, homogeneous or nonhomogeneous. If, as is the case for the Euler equations, a change of variable transforms them into this type of equations, they will not cause any problem. If the equation is of convenient type, CONVODE will be happy to show that, with such a change of variable, your equation becomes an equation with constant coefficients, and will solve it. Do not forget that CONVODE checks if the degree of your equation cannot be simply reduced.

1.5. The generalization of the Euler equations in the case of second-order equations

Some differential linear equations of second order can be transformed to equations with constant coefficients. It is possible if $p(x)$ and $q(x)$ satisfy a well-determined relation. In that case, CONVODE determines the change of variable to perform, and proposes the equation with constant coefficients which results from this change. CONVODE can then go on without any problem.

1.6. The exact second-order equations

In the case where the proposed equation is a second-order equation, CONVODE verifies if this one would not be exact. If it is exact, it is possible to transform it into a first-order equation which can be studied again by CONVODE. Two possibilities are considered: if the equation is exact and linear, the reduction to a first-order equation (linear in this case) is executed by CONVODE and its solution is calculated automatically; if the equation is exact but nonlinear, CONVODE calculates the first-order reduced equation. This equation uses a function $H(X, Y)$ which has to satisfy an equation with partial derivative given by our program. If $H(X, Y)$ is easily determined, then the reduction is complete. ($H(X, Y)$ is the solution of the partial differential equation.)

1.7. The actual possibilities of CONVODE in the general case

In the case of a *linear* differential equation with *nonconstant* coefficients, what can CONVODE do? It is well known that it is not an Euler equation or a second-order exact equation or an equation which can be transformed into an equation with constant coefficients.

- The equation is of second order and you can give a particular solution of the homogeneous equation. In that case, CONVODE finds the second solution and proposes the general solution.
- The equation is of second order and you know nothing about the solution. CONVODE then starts searching a *polynomial* solution with degree inferior to N_{max} (you will have to give an entire value for N_{max}). If a polynomial solution is found and if the order of the polynomial does

not exceed a certain value *ARGSTOP*, then you will find the general solution of your equation. If the solution is a polynomial with a degree greater than *ARGSTOP*, you will have in RESPONSE only the polynomial solution. (This has been developed for people who are only interested in polynomial solutions, and also to avoid calculations in the case where the first solution is a polynomial with too high degree.) There is no objection to solve the equations of Legendre, Laguerre, Hermite, Jacobi, etc.

Attention! CONVODE solves the differential equation, but you have to recognize the polynomial proposed as solution! It is evident that the parameter n associated to the degree of the polynomial, which is in the differential equation, must have a numerically fixed entire value. (CONVODE solves a well-determined equation, but not a class of equations.) We should notice that, if for instance you try to solve Legendre with $n = 0$, CONVODE informs you that the order of your equation can be reduced.

• The differential equation is of order greater than one. In that case, CONVODE only searchs for a polynomial solution of degree smaller than the N_{max} fixed.

1.8. The degenerated systems

We close this introduction to CONVODE with *degenerated* systems. A system of differential equations with constant coefficients is degenerated when the solutions are not linearly independent, that is, there are some contraints between the unknowns. Here the *recursive* character of the program can appear. The system (of first order, or of order greater than one, but with constant coefficients) being detected as degenerated, CONVODE tries to determine a first constraint and, having found this one, the starting system can be reduced (and the constraints can be taken into account). CONVODE starts again with a smaller system and keeps in memory the first constraint found. If the reduced system is not degenerated, then CONVODE finds the solution, adds the constraints and gives the general solution. If the reduced system is still degenerated, a second constraint is found which is added in memory to the first one, the system is reduced for the second time and so on. We can, for example, start with a system of three differential equations of first order where the final solution will involve only one arbitrary constant and in this case two constraints. The number of arbitrary constants which are in the final solution is calculated in the beginning of CONVODE. The system can also be completely degenerated and will not contain constants any more. The degenerated systems are not always compatible, which proves the existence of a constraint impossible to be fulfilled (for instance, $-1 = 0$!). The constraints (if there are some) are stocked in the global variable *NUNU*. In the case of degenerated systems, the initial conditions should be of course compatible with the constraints.

2. How to use CONVODE?

CONVODE is a REDUCE procedure which has five arguments. Each argument is a REDUCE list. (It is always a list to simplify the problems.) `L1:={......};` is the list which contains the equation(s) to be solved. Each element of `L1` must be an equation. `L2:={......};` is the list of the unknowns. `L3:={...};` is the list which contains the independent variable. `L4` is an empty list (`L4:={};`) if the initial conditions are not specified (the solution then uses arbitrary constants), or a list where the last element indicates where the initial conditions are given, the first element of `L4` being the list of those conditions.

Example. The list of the unknowns is `L2:={x, y, z};`. If we want to find the solution which verifies in $t=0$: $x(0)=5$, `df(x, t)=6`, $y(0)=15$, `df(y, t)=67`, `df(y, t, 2)=45` and $z(0)=11$, `df(z, t)=59`, then the list `L4` is `L4:={{5, 6, 15, 67, 45, 11, 59}, 0};`

`L5` is an empty list (`L5:={};`) or `L5:={Francais}`, in which case CONVODE comments in French; `L5:={German}`, etc. The comments are then in the chosen language. The user should not forget to define the global variables `LPARTI:={};` and `NMAX:=1;` Do not forget to introduce the necessary dependence(s).

For example, `DEPEND X, T;`, etc. You only have to call the CONVODE procedure:

```
CONVODE(L1, L2, L3, L4, L5);
```

The result that CONVODE proposes is in the identifier RESPONSE which can be used for checking.

3. Example of some equations solved by CONVODE

CONVODE has been tested on about 1000 equations and systems of equations. I explain hereupon some cases solved with success. The reader than recognize the first-order equations (linear, exact, homogeneous) which admit some integrating factor, separable, of the Bernouilli form, Riccati, Lagrange–Clairaut or first-order equation not solved with respect to the derivative, and so on. I also show some systems of equations (degenerated or not). Then, the Euler equations, the second-order equations with constant coefficients, the exact second-order equations, ending with equations which admit a polynomial solution and systems of order greater than one.

– First-order linear differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y \cos x = \cos x. \tag{1}$$

– Case where the integration reduces to the computation of a simple integral:

$$\sqrt{y}\,\frac{\mathrm{d}y}{\mathrm{d}x} - y^3 = 0. \tag{2}$$

– Example where the equation is exact:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(y - \frac{\sin^2 x}{y^2}\right) + \frac{2 \sin x \cos x}{y} + x = 0. \tag{3}$$

– Factor integrand TESTINT1:

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + x^2 + y^2 + x = 0. \tag{4}$$

– Bernouilli equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\mathrm{e}^x = 2y^{1/2}\mathrm{e}^x. \tag{5}$$

– Factor integrand TESTINT2:

$$2xy^4e^y + 2xy^3 + y + \left(x^2y^4e^y - x^2y^2 - 3x\right)\frac{dy}{dx} = 0. \tag{6}$$

– Case where INTFACT3 is an integrand factor:

$$y(2xy+1) + x\left(1 + 2xy - x^3y^3\right)\frac{dy}{dx} = 0. \tag{7}$$

– Search for an integrand factor of the form X**AL*Y**BE:

$$4xy + 3y^4 + \left(2x^2 + 5xy^3\right)\frac{dy}{dx} = 0. \tag{8}$$

– The separable equations:

$$\sqrt{1+x^2}\,y\frac{dy}{dx} + x\sqrt{1+y^2} = 0. \tag{9}$$

– Homogeneous equations:

$$x^2y + 2xy^2 - y^3 - \left(2y^3 - xy^2 + x^3\right)\frac{dy}{dx} = 0. \tag{10}$$

– Change of variables of the type $X = U + H$, $Y = V + K$:

$$x - y - 1 + (x + 4y - 6)\frac{dy}{dx} = 0. \tag{11}$$

– Is the system for H and K compatible?:

$$x + y + 1 + (2x + 2y - 1)\frac{dy}{dx} = 0. \tag{12}$$

– Transformation of the type Y = ZZ*AL:

$$3y^2\left(y^3 - x\right)\frac{dy}{dx} + x + y^3 = 0. \tag{13}$$

– Except for certain particular cases, it is not possible to solve Riccati. Riccati of class 1 and 2:

$$12\frac{dy}{dx} + 5y^2 = 34x^{-8/5}, \tag{14}$$

$$32\frac{dy}{dx} + 15y^2 = 27x^{-8/3}. \tag{15}$$

– Lagrange–Clairaut (the solution is found under parametric form):

$$y = 2x\frac{dy}{dx} + \log\left(\frac{dy}{dx}\right). \tag{16}$$

– Equation independent of x:

$$y = \left(\frac{dy}{dx} - 1\right)e^{dy/dx}. \tag{17}$$

– Equation independent of y:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 x = \exp\left\{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}\right\}. \tag{18}$$

– Equation nonsolved with respect to the derivative:

$$y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + (x-y)\frac{\mathrm{d}y}{\mathrm{d}x} - x = 0. \tag{19}$$

– System of homogeneous equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 6z, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + 11z, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = -y + 6z. \end{cases} \tag{20}$$

– System of nonhomogeneous equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x + 3y - \cos t, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x - y + \mathrm{e}^{2t}. \end{cases} \tag{21}$$

– One differential equation:

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + 3\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4y = t\mathrm{e}^{-2t}. \tag{22}$$

– System involving derivatives greater than one:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} + 2x + \dfrac{\mathrm{d}y}{\mathrm{d}t} + y = 10 \cos t, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} - 4x + \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3y = 0. \end{cases} \tag{23}$$

– Symbolic coefficients:

$$\begin{cases} \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\dfrac{\mathrm{d}y}{\mathrm{d}t} + n^2x = 0, \\ \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2k\dfrac{\mathrm{d}x}{\mathrm{d}t} + n^2y = 0. \end{cases} \tag{24}$$

– The Euler equation:

$$121\frac{\mathrm{d}^2y}{\mathrm{d}x^2}x^2 - 44\frac{\mathrm{d}^2y}{\mathrm{d}x^2}x + 4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 33\frac{\mathrm{d}y}{\mathrm{d}x}x + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = x. \tag{25}$$

– Second-order equations (Euler generalized):

$$\frac{1}{(e^x+xe^x+2x)^2}\frac{d^2y}{dx^2}-\frac{\left(2e^x+2+xe^x+(e^x+xe^x+2x)^2\right)}{(e^x+xe^x+2x)^3}\frac{dy}{dx}-2y=0. \tag{26}$$

– Exact second-order equations:

$$4x^2\frac{d^2y}{dx^2}-\frac{d^2y}{dx^2}+9x^2\frac{dy}{dx}+18xy-8y=x. \tag{27}$$

– Here is an example where the equation satisfied by y is nonlinear:

$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2+y\frac{dy}{dx}=0. \tag{28}$$

– The CONVODE possibilities in the general case. A case where the solution is entirely determined by CONVODE:

$$(3x+2x^2)\frac{d^2y}{dx^2}-6(1+x)\frac{dy}{dx}+6y=6. \tag{29}$$

– Second-order equation: case where a particular solution of the homogeneous equation is given (`LPARTI:={Y1=COS(E**(_X))}`):

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+e^{-2x}y=\frac{1}{e^{-3x}}. \tag{30}$$

– Differential equations of order greater than 2 with a polynomial solution:

$$x^2\frac{d^4y}{dx^4}+5x\frac{d^3y}{dx^3}+\frac{dy}{dx}\left(2x(al+n+2r)-x^2-al^2+4\right)$$
$$+3\frac{dy}{dx}(al+n+2r-x)+n(n+2)y=0. \tag{31}$$

– A classical polynomial, for instance Gegenbauer ($n=3$). Note that the parameter ν is arbitrary:

$$(1-x^2)\frac{d^2y}{dx^2}-(2\nu+1)x\frac{dy}{dx}+n(2\nu+n)y=0. \tag{32}$$

– The study of degenerated systems (the presence of constraints):

$$\begin{cases}\dfrac{d^2x}{dt^2}+3x+2\dfrac{dy}{dt}=2e^{-t},\\[2ex] \dfrac{dx}{dt}-x+y=-e^{-t}.\end{cases} \tag{33}$$

– Here is an example where the solution does not use arbitrary constants. The problem is completely degenerated:

$$\begin{cases}\dfrac{d^2x}{dt^2}+\dfrac{dx}{dt}+x+\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}+y=t^2,\\[2ex] \dfrac{dx}{dt}+x+\dfrac{dy}{dt}-y=e^t.\end{cases} \tag{34}$$

4. Conclusions and remarks

The CONVODE program has been tested on hundreds of examples of equations and systems of differential equations. A large part of examples come from works given in the references. The CONVODE possibilities are based on the ones of REDUCE and more particularly on the SOLVE and INT functions. It is evident that such a program is interesting when an analytical solution has a chance to be found. If the systems of equations to be solved have too large dimensions, methods of numerical analysis have to be used instead of algebraic methods. Our program works on a VAX 6220 with the REDUCE 3.3 version.

Note added in proof

The last version is now running on a DEC station with REDUCE 3.4.1.

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